AMM Designs Beyond Constant Functions

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Automated Market Makers

Constant Function Market Makers

- A pool with assets X and Y
- Available liquidity x and y
- Deterministic trading function $f(x, y)$

  $\Longrightarrow$ defines the state of the pool before and after a trade
- Liquidity providers (LPs) deposit assets in the pool.

  Liquidity takers (LTs) trade with the pool.

Liquidity takers in a CFMM

LT trading condition

Liquidity Takers

- LTs send a quantity Δy of Y. They receive a quantity Δx of X given by the trading function

$$\underbrace{f(x,y) = f(x - \Delta x, y + \Delta y)}_{\text{LT trading condition}} = \underbrace{\kappa^2}_{\text{Depth}} ,$$

- Level function

$$f(x,y) = \kappa^2 \iff x = \varphi(y)$$

- Execution and instantaneous exchange rates

$$\frac{\Delta x}{\Delta y} \xrightarrow{\Delta y \longrightarrow 0} \underbrace{-\varphi'(y) \equiv Z}_{\text{Instantaneous rate}}$$

- Constant Product Market Makers (CPMMs):

$$f(x,y) = x \times y \quad \text{and} \quad Z = x/y.$$

Liquidity providers in a CFMM

LP trading condition

LP trading condition

- LPs change the depth:

$$f\,(x+\Delta x, y+\Delta y) = K^2 > f\,(x,y) = \kappa^2\,.$$

- LP trading condition: LP does not change the rate:

$$Z = -\varphi^{\kappa\prime}(y) = -\varphi^{K\prime}(y+\Delta y)$$

- LPs hold a portion of the pool and earn fees.

LP trading condition

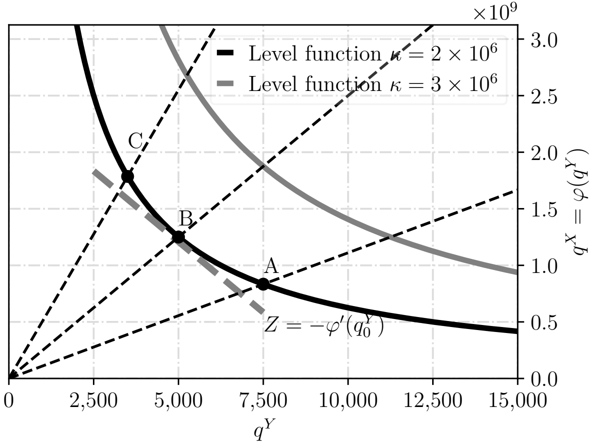

Figure: Geometry of CPMMs: level function $\varphi\left(q^Y\right) = q^X$ for two values of the pool depth.

LP trading condition

In CPMMs

- LP trading condition:

$$\frac{x + \Delta x}{y + \Delta y} = \frac{x}{y}$$

- Depth variations

$$K^2 = (x + \Delta x)(y + \Delta y) > \kappa = x\,y$$

Automated Market Makers Designs Beyond Constant Functions

This talk: arithmetic liquidity pool (ALP).
For more: see the paper where we study the geometric liquidity pool (GLP) too.

ALP

Generalising CFMs:

- A pool receives buy and sell orders with (minimum) size ζ.
- The pool offers liquidity with a spread $[\delta^b, \delta^a]$.
- The dynamics of the reserves:

$$
\begin{aligned}
\mathrm{d}y_t &= \zeta \, \mathrm{d}N_t^b - \zeta \, \mathrm{d}N_t^a \,, \\
\mathrm{d}x_t &= -\zeta \, \left(Z_{t^-} - \delta_t^b\right) \, \mathrm{d}N_t^b + \zeta \left(Z_{t^-} + \delta_t^a\right) \mathrm{d}N_t^a \,.
\end{aligned}
$$

- The dynamics of the price

$$
\mathrm{d}Z_t = -\,\eta^b(y_{t^-}) \, \mathrm{d}N_t^b + \eta^a(y_{t^-}) \, \mathrm{d}N_t^a \,,
$$

for impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$.

ALP

- The reserves take finitely many values $\{\underline{y}, \underline{y}+\zeta, \ldots, \overline{y}\}$.

To simplify notation, let $\mathfrak{y}_1 = \underline{y}$, $\mathfrak{y}_2 = \underline{y}+\zeta$, . . . , and $\mathfrak{y}_N = \overline{y}$ where $N = \overline{N} - \underline{N} + 1$, $\overline{N} = \overline{y}/\zeta$, and $\underline{N} = \underline{y}/\zeta$.

ALP

Theorem:

Let $\varphi(\cdot)$ be the level function of a CFM. Assume one chooses the impact functions

$$\eta^a(y) = \varphi'(y) - \varphi'(y-\zeta)\,, \qquad \eta^b(y) = -\varphi'(y) + \varphi'(y+\zeta)\,,$$

and chooses the quotes

$$\delta_t^a = \frac{\varphi(y_{t-} - \zeta) - \varphi(y_{t-})}{\zeta} + \varphi'(y_{t-})\,, \tag{1}$$

$$\delta_t^b = \frac{\varphi(y_{t-} + \zeta) - \varphi(y_{t-})}{\zeta} - \varphi'(y_{t-})\,. \tag{2}$$

Then ALP $\equiv$ CFM.

Arbitrage?

Theorem:

Under certain reasonable conditions on the impact functions η^a and η^b (see the paper) then there is no roundtrip sequence of trades that a liquidity taker can execute to arbitrage the ALP.

Arbitrage?

An example of a “reasonable” condition.

A nice class of impact functions

Proposition:

The marginal rate Z takes only the ordered finitely many values $\mathcal{Z} = \{\mathfrak{z}_1, \ldots, \mathfrak{z}_N\}$, with the property that $Z_0 \in \mathcal{Z}$ and for $i \in \{1, \ldots, N-1\}$

$$\mathfrak{z}_{i+1} - \eta^b(\mathfrak{y}_{N-i}) = \mathfrak{z}_i \qquad \text{and} \qquad \mathfrak{z}_i + \eta^a(\mathfrak{y}_{N-i} + \zeta) = \mathfrak{z}_{i+1}\,, \tag{3}$$

if and only if $\eta^a(\cdot)$ and $\eta^b(\cdot)$ are such that

$$\eta^b(\mathfrak{y}_i) = \eta^a(\mathfrak{y}_i + \zeta)\,, \tag{4}$$

for $i \in \{1, \ldots, N-1\}$.

Arbitrage?

So far we have only discussed the mechanics of our framework, which is general enough to have CFMs as a subset. So, let's write a model to underpin the new design.

A new design

The LP models the intensity of order arrivals as:

$$\begin{cases} \lambda_t^b\left(\delta_t^b\right) = c^b\, e^{-\kappa\, \delta_t^b}\, \mathbb{1}^b\left(y_{t-}\right)\,, \\ \lambda_t^a\left(\delta_t^a\right) = c^a\, e^{-\kappa\, \delta_t^a}\, \mathbb{1}^a\left(y_{t-}\right)\,, \end{cases} \tag{5}$$

where c^a and c^b are two non-negative constants that capture the baseline selling and buying intensity, respectively, and where

$$\mathbb{1}^b(y) = \mathbb{1}_{\{y+\zeta\leq\overline{y}\}} \quad \text{and} \quad \mathbb{1}^a(y) = \mathbb{1}_{\{y-\zeta\geq\underline{y}\}}\,, \tag{6}$$

indicate that the ALP stops using the LP's liquidity upon reaching her inventory limits $\underline{y}, \overline{y}$.

A new design

- The LP chooses the impact functions η^b and η^a.
- The admissible set of strategies is given by all squared-integrable, adapted, bounded from below δ^a, δ^b.
- For the price of liquidity $\{\delta^b, \delta^a\}$: the performance criterion using $\delta = \left(\delta^b, \delta^a\right)$ is the function w^δ:

$$w^\delta(t, x, y, z) = \mathbb{E}_{t,x,y,z}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2\, \mathrm{d}s\right] ,$$

- We wish to find $\delta^* = \arg\max_\delta w^\delta(0, x, y, z)$
- Closed-form solution!
- In our design: CFMs are **suboptimal**.

CFMs are suboptimal

Proposition:

Let $\varphi(\cdot)$ be the level function of a CFM. Consider an LP with initial wealth (x_0, y_0) who sets a liquidity posititon in the CFM and whose performance criterion is given by

$$J^{\text{CFM}} = \mathbb{E}\left[x_T^{\text{CFM}} + y_T^{\text{CFM}}\, Z_T^{\text{CFM}} - \alpha\,(y_T^{\text{CFM}} - \hat{y})^2 - \phi \int_0^T (y_s^{\text{CFM}} - \hat{y})^2 \,\mathrm{d}s\right], \tag{7}$$

with $J^{\text{CFM}} \in \mathbb{R}$. Consider an LP in a ALP with initial wealth (x_0, y_0) and with impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$ given by the ones that match the dynamics of a CFM. Let $\delta_t^{CFM} = \left(\delta_t^{a,\text{CFM}}, \delta_t^{b,\text{CFM}}\right)$ be given by the distances that match those in a CFM.

CFMs are suboptimal

Consider the performance criterion $J : \mathcal{A}_0 \to \mathbb{R}$

$$J(\delta) = \mathbb{E}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_0^T (y_s - \hat{y})^2\,\mathrm{d}s\right] , \qquad (8)$$

where $\delta = (\delta^a,\, \delta^b)$ is an admissible strategy. Then,

$$J^{\text{CFM}} = J\left(\delta^{CFM}\right) \qquad \text{and} \qquad J^{\text{CFM}} \leq J\left(\delta^{\star}\right) , \qquad (9)$$

where $\delta^{\star} = (\delta^{a,\star}, \delta^{b,\star})$ is the optimal strategy we find (in closed-form!).

A sneak peek at the optimal strategy

The optimal strategy in feedback form is

$$\delta^{b\star}(t, y_{t^-}) = \frac{1}{\kappa} - \frac{\theta\,(t, y_{t^-} + \zeta) - \theta\,(t, y_{t^-})}{\zeta} - \frac{(y_{t^-} + \zeta)\,\eta^b(y_{t^-})}{\zeta}\,, \quad (10)$$

$$\delta^{a\star}(t, y_{t^-}) = \frac{1}{\kappa} - \frac{\theta\,(t, y_{t^-} - \zeta) - \theta\,(t, y_{t^-})}{\zeta} + \frac{(y_{t^-} - \zeta)\,\eta^a(y_{t^-})}{\zeta}\,, \quad (11)$$

for a function $\theta(\cdot)$ we find in closed-form.

The new design in a little more detail

Our theorem states what δ^* (price of liquidity) is once $\eta^a(\cdot), \eta^b(\cdot)$ and model parameters (e.g. $\alpha, \phi, \hat{y}$) are specified.

The new design asks that LPs specify their impact functions and model parameters and the “venue” plays by the rules imposed by the dynamics and the optimal strategy.

Numerical implementation

In the numerical examples below, we assume that $c^a = c^b = c > 0$ and that the inventory risk constraint is $y \in \{\underline{y}, \ldots, \overline{y}\}$ where $\underline{y} \geq \zeta$. Then, we employ the following impact functions for the liquidity provision strategy in the ALP:

$$\eta^b(y) = \frac{\zeta}{\frac{1}{2}\,y + \zeta}\,L \quad \text{and} \quad \eta^a(y) = \frac{\zeta}{\frac{1}{2}\,y - \zeta}\,L\,, \tag{12}$$

where $L > 0$ is the impact parameter.

Numerical implementation

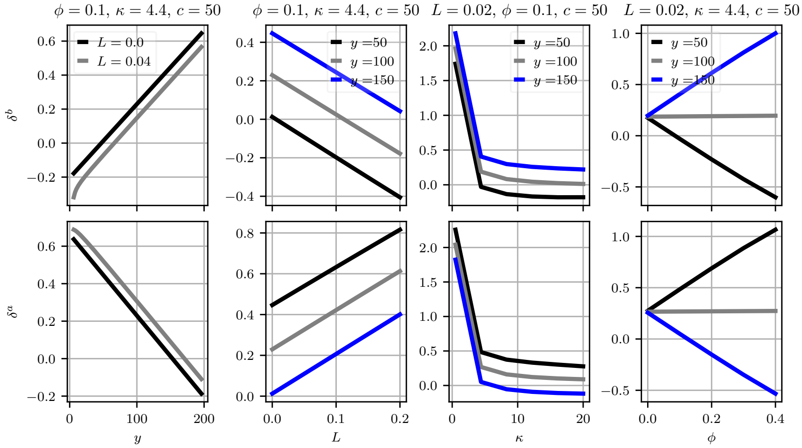

Figure: ALP: Optimal shifts as a function of model parameters, where $\hat{y} = 100$ ETH, $[\underline{y}, \overline{y}] = [\zeta, 200]$, and $\alpha = 0\,\text{USDC} \cdot \text{ETH}^{-2}$.

A new design

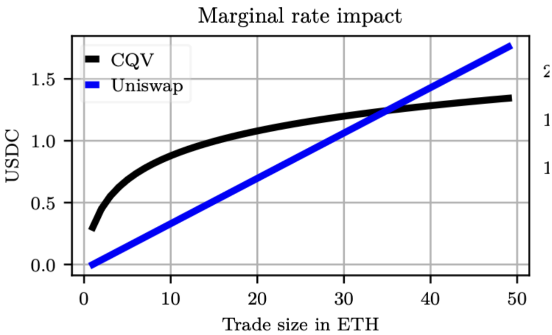

Figure: Marginal rate impact and execution costs in the ALP as a function of the size of the trade.

A new design

	Average	Standard deviation
ALP (scenario I)	-0.004%	0.719%
ALP (scenario II)	0.717%	2.584%
Buy and Hold	0.001%	0.741%
Uniswap v3	-1.485%	7.812%

Table: Average and standard deviation of 30-minutes performance of LPs in the ALP for both simulation scenarios, LPs in Uniswap, and buy-and-hold.

Merci | Thank you