Automated Market Makers: Mean-Variance Analysis of LPs Payoffs and Design of Pricing Functions

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joint work with
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1. Introduction

Automated Market Makers (AMM)

- AMMs are decentralized protocols in which
  - 2 or more assets are maintained as reserves
  - Liquidity Providers (LPs) deposits liquidity in reserves assets
  - Liquidity Takers (LTs) can swap reserve assets against
  - Prices are determined by a pricing rules : CFMM
  - No counterparty risk
- We analyze AMMs from the point of view of LPs
- All CFMMs are subject to impermanent loss :
  - If assets prices are different when a LP enters and exists the pool, she will incur a loss
  - Trading fees are meant to overcome this loss

Our contribution

1. Explore AMMs in which the pricing function uses some information from another trading venue
   $\implies$ quite similar to the situation of a traditional market maker in dealer markets
2. Build in this paper a simple mean-variance framework inspired from the so-called modern portfolio theory, in order to compare different AMMs
3. Estimate the maximum extra return that a LP could expect for a given level of tracking error with respect to Hodl
   $\implies$ efficient market making strategies

Literature Review

- CPMM : Angeris, Kao, Chiang, Noyes and Chitra (2019), Clark (2020), Clark (2021)
- CFMM : Angeris and Chitra (2020), Angeris, Evans and Chitra (2021)
- LPs return : Angeris, Evans and Chitra (2020), Evans (2020)
- General AMM :
  - Optimal fees : Angeris Evans and Chitra (2021), Fritsch, Kaser and Wattenhofer (2022), Hasbrouck, Rivera and Saleh (2022)
  - Strategic LPs : Aoyagi (2020), Cartea, Drissi and Monga (2022), Neuder, Rao, Moroz and Parkes (2021)
  - Execution : Angeris, Evans, Chitra and Boyd (2022), Cartea, Drissi and Monga (2022), Park (2022)
  - Competition : Aoyagi and Ito (2021), Lehar and Parlour (2021), Barbon and Ranaldo (2022)
- Market Making : Ho and Stoll (1981), Ho and Stoll (1983), Avellaneda and Stoikov (2008), Guéant, Lehalle and Fernandez-Tapia (2013), Cartea, Jaimungal and Ricci (2014)

2. Efficient pricing functions in the perfect information case

The model

- We consider a pool of 2 assets.
- $\left(q_t^0\right)_t$ and $\left(q_t^1\right)_t$: reserves in currency 0 and currency 1 in the pool
- We assume the exogenous exchange rate, S_t, to follow a GBM

$$dS_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

- Transaction sizes are labelled in the accounting currency (currency 0)
  - A part corresponding to the market exchange rate
  - A part corresponding to a markup, accounted in currency 0

Example

If a client wants to sell z coins of currency 0 at time t, then z/S_t coins of currency 1 will be offered to her and $z\delta^{0,1}(t,z)$ extra coins of currency 0 will be asked as a markup

The model

Admissible markups

The markups $\left(\delta^{0,1}, \delta^{1,0}\right)$ belong to

$$\mathcal{A} := \left\{ \delta = \left(\delta^{0,1}, \delta^{1,0}\right) : \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto \mathbb{R}^2 \middle| \delta \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*)\text{-measurable} \right.$$
$$\left. \text{and } \delta^{0,1}(t,z) \wedge \delta^{1,0}(t,z) \geq -C \ \mathbb{P} \otimes dt \otimes dz \ a.e. \right\}$$

Markups process and reserves

$$dX_t = \int_{z \in \mathbb{R}_+^*} z\delta^{0,1}(t,z) J^{0,1}(dt,dz) + \int_{z \in \mathbb{R}_+^*} z\delta^{1,0}(t,z) J^{1,0}(dt,dz)$$

$$dq_t^0 = \int_{z \in \mathbb{R}_+^*} z\left(J^{0,1}(dt,dz) - J^{1,0}(dt,dz)\right) \text{ and } dq_t^1 = \int_{z \in \mathbb{R}_+^*} \frac{z}{S_t}\left(J^{1,0}(dt,dz) - J^{0,1}(dt,dz)\right)$$

The model

Transaction processes

The processes $J^{0,1}(dt,dz)$ and $J^{1,0}(dt,dz)$ have known intensity kernels given respectively by $(\nu_t^{0,1}(dz))_{t\in\mathbb{R}_+}$ and $(\nu_t^{1,0}(dz))_{t\in\mathbb{R}_+}$, verifying

$$\nu_t^{0,1}(dz) = \Lambda^{0,1}\left(z,\delta^{0,1}(t,z)\right)\mathbb{1}_{\{q^{\mathbf{1}}_{t-}\geq \frac{z}{S_t}\}}m(dz)$$

and

$$\nu_t^{1,0}(dz) = \Lambda^{1,0}\left(z,\delta^{1,0}(t,z)\right)\mathbb{1}_{\{q^{\mathbf{0}}_{t-}\geq z\}}m(dz),$$

where m is a measure and $\Lambda^{0,1}$ and $\Lambda^{1,0}$ are called the intensity functions of the processes $J^{0,1}(dt,dz)$ and $J^{1,0}(dt,dz)$ respectively

Example

$$\Lambda^{0,1}(z,\delta) = \frac{\lambda^{0,1}(z)}{1+e^{\alpha^{\mathbf{0,1}}(z)+\beta^{\mathbf{0,1}}(z)\delta}} \quad \text{and} \quad \Lambda^{1,0}(z,\delta) = \frac{\lambda^{1,0}(z)}{1+e^{\alpha^{\mathbf{1,0}}(z)+\beta^{\mathbf{1,0}}(z)\delta}}$$

Comparison with Hodl - Excess reserves

We introduce the following two processes

$$\left(Y_t^0\right)_{t\in\mathbb{R}_+} = \left((q_t^0 - q_0^0)\right)_{t\in\mathbb{R}_+} \quad \text{and} \quad \left(Y_t^1\right)_{t\in\mathbb{R}_+} = \left((q_t^1 - q_0^1)S_t\right)_{t\in\mathbb{R}_+}$$

with dynamics

$$dY_t^0 = \int_{z\in\mathbb{R}_+^*} z\left(J^{0,1}(dt,dz) - J^{1,0}(dt,dz)\right)$$

and

$$dY_t^1 = \mu Y_t^1 dt + \sigma Y_t^1 dW_t + \int_{z\in\mathbb{R}_+^*} z\left(J^{1,0}(dt,dz) - J^{0,1}(dt,dz)\right)$$

Comparison with Hodl - Excess PnL

- The excess PnL is defined by the markups process and the terminal excess reserves.

$$
\begin{aligned}
\mathrm{PnL}_T - \mathrm{PnL}_T^{\mathrm{Hodl}} &= X_T + Y_T^0 + Y_T^1 \\
&= \int_0^T \int_{z \in \mathbb{R}_+^*} z\delta^{0,1}(t,z) J^{0,1}(dt,dz) + \int_0^T \int_{z \in \mathbb{R}_+^*} z\delta^{1,0}(t,z) J^{1,0}(dt,dz) \\
&\quad + \int_0^T \mu Y_t^1 dt + \int_0^T \sigma Y_t^1 dW_t
\end{aligned}
$$

A mean-variance analysis

Market simulator

- Currency 0: USD, Currency 1: ETH
- Initial exchange rate: 1600 USD per ETH
- Drift: $\mu = 0$ day^{-1}
- Volatility: $\sigma = 0.052$ day^{-1} (it corresponds to an annualized volatility of 100%)
- Single transaction size: 4000 USD (i.e. *m* is a Dirac mass)
- Parameters of intensity functions: $\lambda^{0,1} = \lambda^{1,0} = 100$ day^{-1}, $\alpha^{0,1} = \alpha^{1,0} = -1.8$, $\beta^{0,1} = \beta^{1,0} = 1300$ bps^{-1}
- Initial inventory: $2,000,000$ USD and $1,250$ ETH
- Time horizon: $T = 0.5$ day

Constant markups vs CPMM

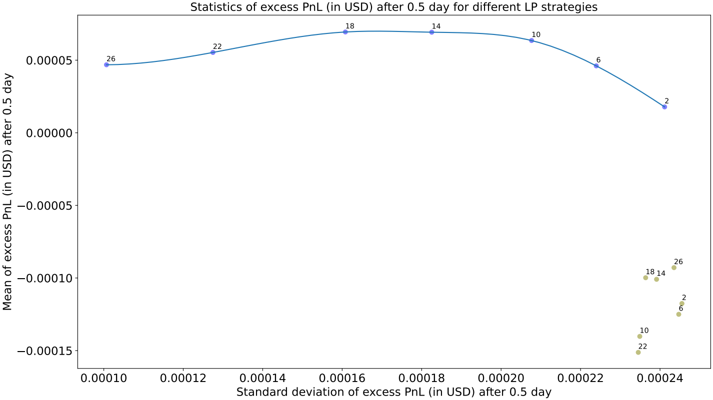

Figure: In blue: naive strategies with constant $\delta^{0,1}, \delta^{1,0}$ (the numbers next to the points correspond to the value of $\delta^{0,1}$ and $\delta^{1,0}$ in bps). In light green: CPMM with transaction fees (the numbers next to the points correspond to the transaction fees in bps).

Optimal Strategies

- Derive the optimal quote under perfect information
- Optimizing the "Mean-quadratic-variation" => with γ the risk aversion coefficient

Objective function

$$\sup_{(\delta^{\mathbf{0},\mathbf{1}},\delta^{\mathbf{1},\mathbf{0}})\in\mathcal{A}} \mathbb{E}\Bigg[\int_0^T \Bigg\{ \int_{z\in\mathbb{R}_+^*} \Big(z\delta^{0,1}(t,z)\Lambda^{0,1}(z,\delta^{0,1}(t,z))\mathbb{1}_{\{q^{\mathbf{1}}_{t-}\geq \frac{z}{S_t}\}}$$

$$+z\delta^{1,0}(t,z)\Lambda^{1,0}(z,\delta^{1,0}(t,z))\mathbb{1}_{\{q^{\mathbf{0}}_{t-}\geq z\}}\Big) m(dz) + \mu Y^1_t - \frac{\gamma}{2}\sigma^2(Y^1_t)^2 \Bigg\} dt\Bigg]$$

Remark

4 state variables $\implies$ numerically intractable

Optimal Strategies

- For moderate values of μ, the quadratic penalty provides an incentive to keep the composition of the pool close to the initial one
- The no-depletion constraints (the indicator functions) can be regarded as superflous

Approximation

$$\sup_{(\delta^{0,1},\delta^{1,0})\in\mathcal{A}} \mathbb{E}\left[\int_0^T \left\{ \int_{z\in\mathbb{R}_+^*} \Big(z\delta^{0,1}(t,z)\Lambda^{0,1}(z,\delta^{0,1}(t,z)) + z\delta^{1,0}(t,z)\Lambda^{1,0}(z,\delta^{1,0}(t,z))\Big) m(dz) + \mu Y_t^1 - \frac{\gamma}{2}\sigma^2 (Y_t^1)^2 \right\} dt\right]$$

Remark

Only one state variable, Y^1, with Markovian dynamics $\implies$ tractable

- It is indeed numerically verified that the reserves remain positive

An optimization problem

HJB equation

$$\begin{cases} 0 = \partial_t\theta(t,y) + \mu y(1+\partial_y\theta(t,y)) - \frac{\gamma}{2}\sigma^2 y^2 + \frac{1}{2}\sigma^2 y^2 \partial^2_{yy}\theta(t,y) \\ \qquad + \int_{\mathbb{R}_+^*}\left(zH^{0,1}\left(z, \frac{\theta(t,y)-\theta(t,y-z)}{z}\right) + zH^{1,0}\left(z, \frac{\theta(t,y)-\theta(t,y+z)}{z}\right)\right) m(dz) \\ \theta(T,y) = 0 \end{cases}$$

Hamiltonian functions

$$H^{0,1}(z,p) = \sup_{\delta \geq -C} \Lambda^{0,1}(z,\delta)(\delta - p) \text{ and } H^{1,0}(z,p) = \sup_{\delta \geq -C} \Lambda^{1,0}(z,\delta)(\delta - p)$$

An optimization problem

Optimal controls

The supremum in the definition of $H^{i,j}(z,p)$ is reached at a unique $\bar{\delta}^{i,j}(z,p)$ given by

$$\bar{\delta}^{i,j}(z,p) = \left(\Lambda^{i,j}\right)^{-1}\left(z, -\partial_p H^{i,j}(z,p)\right)$$

The markups that maximize our modified objective function are obtained in the following form

$$\delta^{0,1*}(t,z) = \bar{\delta}^{0,1}\left(z, \frac{\theta(t, Y^1_{t-}) - \theta(t, Y^1_{t-} - z)}{z}\right)$$

and

$$\delta^{1,0*}(t,z) = \bar{\delta}^{1,0}\left(z, \frac{\theta(t, Y^1_{t-}) - \theta(t, Y^1_{t-} + z)}{z}\right).$$

Efficient frontier

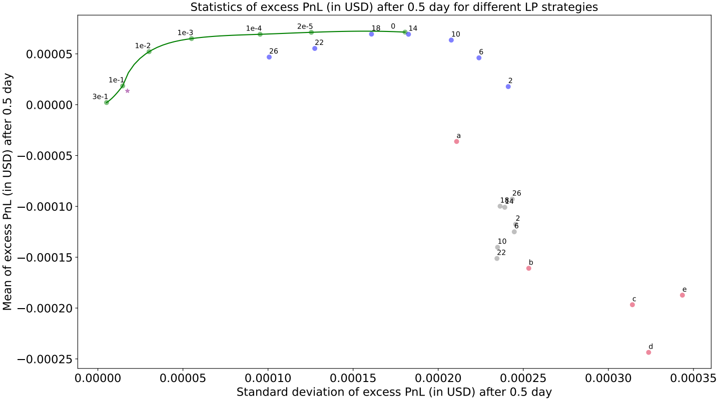

Figure: In blue: naive strategies with constant $\delta^{0,1}, \delta^{1,0}$. In grey: CPMM with fees. In pink: other CFMM without market exchange rate, for different sets of realistic parameters. In purple ($\star$): CFMM with market exchange rate oracle. In green: efficient frontier, obtained using the optimal markups for different levels of risk aversion.

3. Misspecification, partial information and arbitrages

Misspecification

- We evaluate the performance of the optimal quotes if the model parameters differs from the expected one
- 3 main parameters to evaluate :
  - The drift : μ
  - The volatility : σ
  - The liquidity parameters : $\lambda^{i,j}$
- In the HJB, there is a relationship between σ, $\lambda^{i,j}$ and γ, so that a shift in σ or $\lambda^{i,j}$ correspond to a shift in γ

Misspecification: $\lambda^{i,j}$

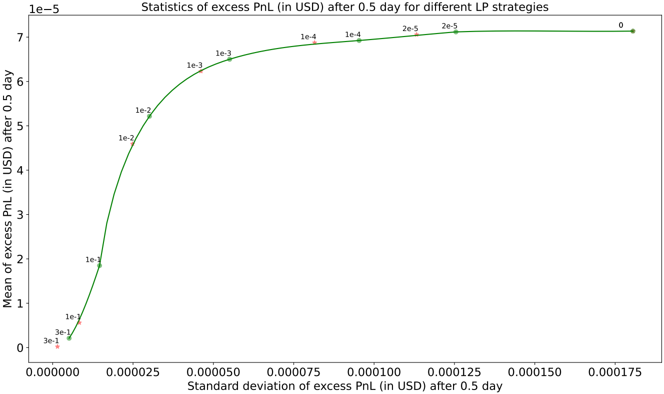

Figure: Performance of strategies in terms of mean / standard deviation of excess PnL when $\lambda^{0,1} = \lambda^{1,0} = 100 \text{ day}^{-1}$. In green: efficient frontier, obtained with the efficient strategy for different levels of risk aversion with perfect information. In pink: performance of the misspecified strategy obtained with $\lambda^{0,1} = \lambda^{1,0} = 50 \text{ day}^{-1}$ for different levels of risk aversion.

Misspecification: σ

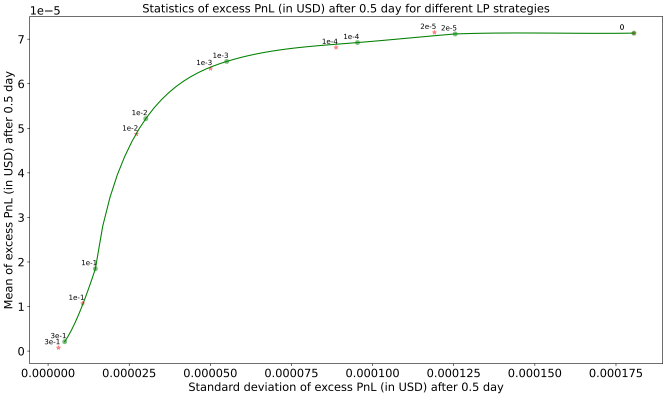

Figure: Performance of strategies in terms of mean / standard deviation of excess PnL when $\sigma = 1$ year^{-1}. In green: efficient frontier, obtained with the efficient strategy for different levels of risk aversion with perfect information. In pink: performance of the misspecified strategy obtained with $\sigma = 1.2$ year^{-1} for different levels of risk aversion.

Misspecification: μ

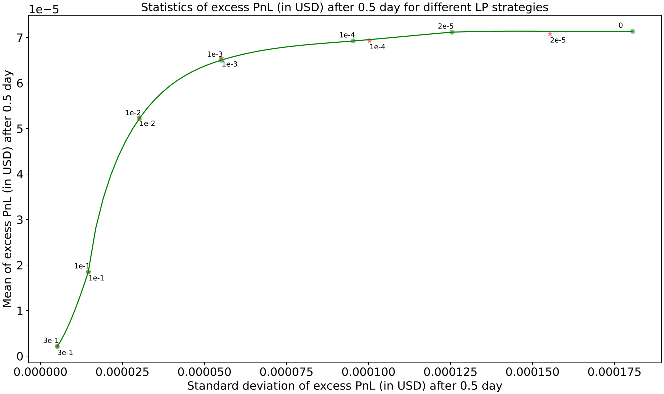

Figure: Performance of strategies in terms of mean / standard deviation of excess PnL when $\mu = 0$. In green: efficient frontier, obtained with the efficient strategy for different levels of risk aversion with perfect information. In pink: performance of the misspecified strategy obtained with $\mu = 0.4$ year^{-1} for different levels of risk aversion.

Partial information

Oracles

- Main limitation of the model: the market exchange rate is assumed to be known at all time
- In practice, we can feed a smart contract with external data through an oracle, but this can only be done at discrete times

Partial information

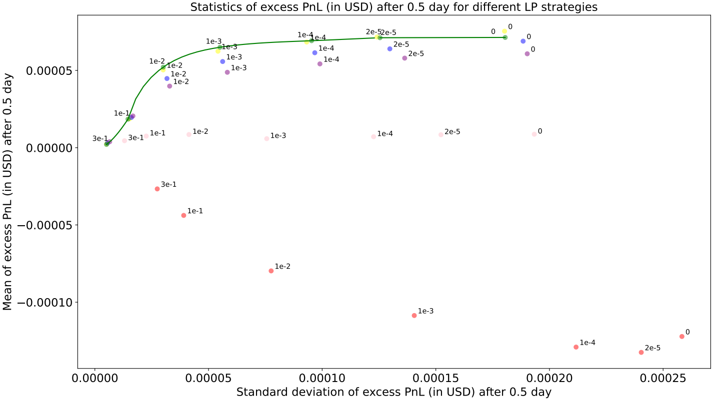

Figure: Performance of the efficient strategies in terms of mean / standard deviation of excess PnL, obtained by playing the efficient strategies for different levels of risk aversion with different oracle delays: perfect information (in green), 10 seconds delay (in yellow), 30 seconds delay (in blue), 1 minute delay (in purple), 5 minutes delay (in pink), 30 minutes delay (in red).

Arbitrage

- Partial information regarding the market exchange rate can sometimes result in arbitrage opportunities for LTs
- Already taken into account in the demand curves modeled by the intensity functions, though not in a systematic way: if a price appears to be very good for LTs, the probability that a transaction occurs is very high
- In practice however, there exists a category of agents, called arbitrageurs, who systematically exploit arbitrage opportunities: they trade with the AMM until arbitrage opportunities disappear

Arbitrages

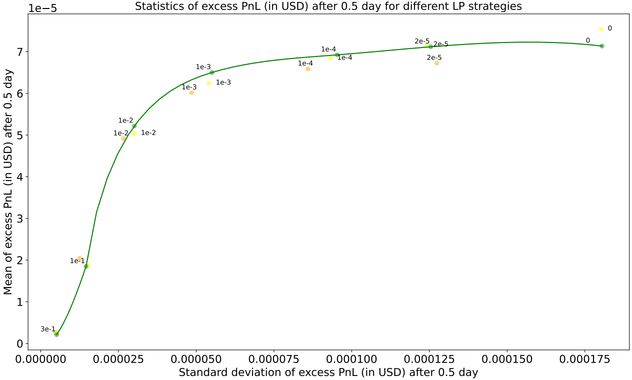

Figure: In green: efficient frontier, obtained by playing the optimal strategy for different levels of risk aversion with perfect information. In yellow: performance of the same optimal strategy for different levels of risk aversion with a discrete oracle. In orange: performance of the same optimal strategy for different levels of risk aversion with a discrete oracle and with arbitrage flow.

Arbitrages

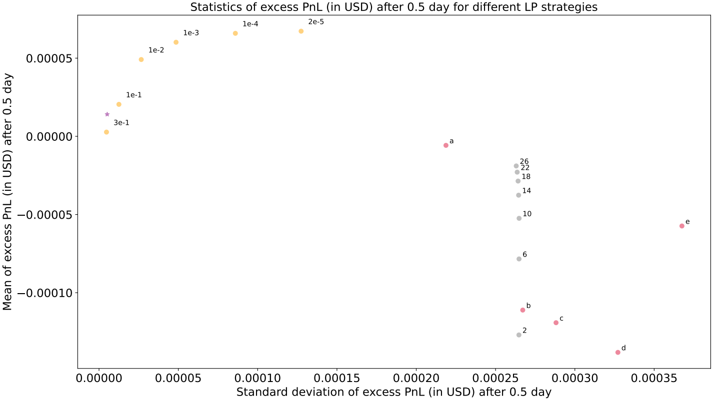

Figure: In grey: CPMM with fees. In pink: CFMM without market exchange rate oracle for different sets of realistic parameters. In purple ($\star$): CFMM with market exchange rate oracle. In orange: performance of the efficient strategies for different levels of risk aversion.

4. Conclusion

Conclusion

- Traditional CFMMs perform poorly relative to the efficient frontier and very often exhibit negative excess PnL
- Allowing an AMM to get information about the current market exchange rate (through an oracle) can significantly improve performance $\implies$ significantly reduces the volatility of the excess PnL while delivering a positive excess PnL on average
- Introducing an oracle in the AMM design comes at the cost that the oracle itself should be carefully designed to avoid introducing additional attack vectors

Thank You !

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